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On some results of Cufaro Petroni about Student t-processes

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Abstract
This paper deals with Student t-processes as studied in Cufaro Petroni (2007 *J. Phys. A. Math. Theor.* **40** 2227–50). We prove and extend some conjectures expressed by Cufaro Petroni about the asymptotical behavior of a Student t-process and the expansion of its density. First, the explicit asymptotic behavior of any real positive convolution power of a Student t-density with any real positive degrees of freedom is given in the multivariate case; then the integer convolution power of a Student t-distribution with odd degrees of freedom is shown to be a convex combination of Student t-densities with odd degrees of freedom. At last, we show that this result does not extend to the case of non-integer convolution powers.

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1. Introduction

In a recent contribution (Cufaro Petroni 2007), Cufaro Petroni derived several results about the behavior of some non-stable Lévy processes with Student t-marginals and random walks with Student t-increments. We recall that the Student t-density with $f = 2\nu$ degrees of freedom $(\nu > 0)$ is[3]

$$p_\nu(x) = A_\nu(1+x^2)^{-(\nu+\frac{1}{2})}, \qquad A_\nu = \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(\nu)}.$$

The family of Student t-densities includes the Cauchy density for $f = 1$ and the scaled density $\frac{1}{\sqrt{2\nu}}p_\nu\left(\frac{x}{\sqrt{2\nu}}\right)$ converges to the Gaussian density as $f \to +\infty$. All Student t-distributions

[3] Usually, the Student t-distribution is defined by the scaled version $\frac{1}{\sqrt{2\nu}}p_\nu(\frac{x}{\sqrt{2\nu}})$, see Feller (1950, p 49).

are heavy tailed. Grosswald (1976) proved that they are infinitely divisible. They also have the stronger property of being self-decomposable, cf Steutel and van Harn (2004).

Stochastic processes with Student t-marginals and various types of dependence structures have been proposed in Heyde and Leonenko (2005), most of them with dependent increments. On the other side, Cufaro Petroni's paper deals with Lévy Student t-processes, which exist by the infinite divisibility of the Student t-distribution. In both cases, these processes have applications in finance (Schoutens 2003) and physics (Vivoli *et al* 2006).

2. Three conjectures by Cufaro Petroni

Let us consider the random walk

$$Z_N = \sum_{i=1}^{N} X_i,$$

where $N \in \mathbb{N}$ and each independent step X_i follows a Student t-distribution with $f = 2n + 1$ degrees of freedom, $n \in \mathbb{N}$. Cufaro Petroni obtained precise results about the process Z_N only in the case of $f = 3$ degrees of freedom; however, he expressed three conjectures about the extension of these results to more general cases: the first conjecture is

Conjecture 1. *For all $N \in \mathbb{N}$ and for all $f = 2n + 1, n \in \mathbb{N}$, the distribution of $N^{-1}Z_N$ is a convex combination of Student t-distributions with odd degrees of freedom.*

The two remaining conjectures concern the distribution of Z_N for non-integer values of N, which makes sense because the Student t-distribution is infinitely divisible: the c-fold convolution of the distribution p_ν is defined, for any real positive c, as the inverse Fourier transform

$$p_\nu^{*c}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}ux} [\varphi_\nu(u)]^c \,\mathrm{d}u,$$

where $\varphi_\nu(u)$ is the characteristic function of the Student t-distribution

$$\varphi_\nu(u) = k_\nu(|u|)$$

with

$$k_\nu(u) = \frac{2^{1-\nu}}{\Gamma(\nu)} u^\nu K_\nu(u), \qquad u > 0. \tag{1}$$

Here K_ν is the modified Bessel function of the second kind also called the Macdonald function. Expression (1) reduces to elementary functions exactly when $\nu = n + 1/2, n = 0, 1, \ldots$ because

$$k_{n+\frac{1}{2}}(u) = \mathrm{e}^{-u} q_n(u), \qquad u > 0, \tag{2}$$

where q_n is a polynomial of degree n with positive coefficients, called the nth Bessel polynomial. It is given as

$$q_n(u) = \sum_{k=0}^{n} \alpha_k^{(n)} u^k, \tag{3}$$

where

$$\alpha_k^{(n)} = \frac{(-n)_k 2^k}{(-2n)_k k!}. \tag{4}$$

The first examples of these polynomials are

$$q_0(u) = 1, \qquad q_1(u) = 1 + u, \qquad q_2(u) = 1 + u + \frac{u^2}{3}.$$

Those properties and further (including historical) information on Bessel polynomials can be found in Berg and Vignat (2008), Gálvez and Dehesa (1984) and the references therein.

Cufaro Petroni's second conjecture concerns the asymptotic behavior of the density of the c-fold convolution of p_ν.

Conjecture 2. *For every $c > 0$ and $\nu > 0$, the asymptotic behavior of the c-fold convolution p_ν^{*c} is given by*

$$p_\nu^{*c}(x) \sim \frac{cA_\nu}{x^{2\nu+1}}, \qquad x \to +\infty.$$

Cufaro Petroni's last conjecture is an extension of conjecture 1 to the c-fold convolution p_ν^{*c} as follows.

Conjecture 3. *Conjecture 1 extends to non-integer sampling times c under the following form: for all $\nu_0 > 0$ and all $c > 0$,*

$$p_{\nu_0}^{*c}(x) = \int_{\nu_0}^{+\infty} \frac{1}{c} p_\nu \left(\frac{x}{c}\right) Q_{\nu_0,c}(\mathrm{d}\nu)$$

for some distribution $Q_{\nu_0,c}(\nu)$.

In this paper, we show that conjecture (1) holds true and give an extended version of it; likewise, we prove conjecture (2). We were unable to prove or disprove conjecture (3), but we disprove a discrete version of it in the case where $\nu = n + \frac{1}{2}$ with $n \in \mathbb{N}$.

Moreover, we consider in the rest of this paper the multivariate context: all Student t-variables are supposed rotation invariant d-dimensional vectors. The multivariate Student t-density is given, for $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ by

$$p_\nu(\mathbf{x}) = A_{d,\nu}(1 + |\mathbf{x}|^2)^{-\nu-d/2}, \qquad A_{d,\nu} = \frac{\Gamma\left(\nu + \frac{d}{2}\right)}{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)^d},$$

where

$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^d x_i y_i, \qquad |\mathbf{x}| = \langle \mathbf{x}, \mathbf{x}\rangle^{\frac{1}{2}}, \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

3. First conjecture: the odd degrees of freedom case

Cufaro Petroni's first conjecture (Cufaro Petroni 2007, proposition 5.2) is that if X_i is a set of independent Student t-distributed random variables with $f = 2n + 1, n \in \mathbb{N}$ degrees of freedom, then the density of the distribution of the normalized Nth step of the random walk

$$N^{-1} Z_N = \frac{1}{N} \sum_{i=1}^N X_i$$

is written as

$$v(x) = \sum_{k=n}^{nN} \beta_k^{(n,N)} p_{k+\frac{1}{2}}(x) \tag{5}$$

with $\beta_k^{(n,N)} \geqslant 0, n \leqslant k \leqslant nN$. We extend and prove this conjecture as follows.

Theorem 1. *If $N \in \mathbb{N}$ and*

$$\mathbf{Y}_N = \sum_{i=1}^{N} a_i \mathbf{X}_i,$$

where a_i are positive numbers with sum 1 and $\mathbf{X}_i$ are independent d-variate Student t-distributed, each with $f_i = 2n_i + 1 (n_i \in \mathbb{N})$ degrees of freedom, then the density of $\mathbf{Y}_N$ is

$$\sum_{j=min(n_1,\ldots,n_N)}^{n_1+\ldots+n_N} \beta_j p_{j+\frac{1}{2}}(\mathbf{x}),$$

where the coefficients β_j are non-negative with sum 1 and depend on N and on coefficients $a_1, \ldots, a_N$ and $n_1, \ldots, n_N$ but not on the dimension d.

Proof. The characteristic function of the d-variate Student t-distribution is

$$\varphi_\nu(\mathbf{u}) = k_\nu(|\mathbf{u}|)$$

where the function k_ν is given by (1). Since for $\nu_i = n_i + \frac{1}{2}$, this function reads $k_{\nu_i}(|\mathbf{u}|) = \mathrm{e}^{-|\mathbf{u}|} q_{n_i}(|\mathbf{u}|)$ where q_{n_i} is the Bessel polynomial of degree n_i, the result follows from Berg and Vignat (2008, theorem 2.6):

$$q_{n_1}(a_1 u) q_{n_2}(a_2 u) \cdots q_{n_N}(a_N u) = \sum_{j=l}^{L} \beta_j q_j(u), \qquad u \in \mathbb{R} \tag{6}$$

with non-negative coefficients β_j with sum 1 and $l = \min(n_1, \ldots, n_N)$, $L = n_1 + \cdots + n_N$. □

As a particular case, choosing $a_i = \frac{1}{N}, 1 \leqslant i \leqslant N$ for $N \in \mathbb{N}$, we deduce that the coefficients $\beta_k^{(n,N)}$ in (5) are positive, and thus the density v is a convex combination of Student t-distributions with odd degrees of freedom.

We are not able to provide an expression for the coefficients $\beta_k^{(n,N)}$ which can be used directly to see the non-negativity. Using Carlitz' formula, see Berg and Vignat (2008),

$$u^n = \sum_{j=0}^{n} \delta_j^{(n)} q_j(u), \qquad n = 0, 1, \ldots \tag{7}$$

with

$$\delta_j^{(n)} = \begin{cases} \dfrac{(n+1)!}{2^n} \dfrac{(-1)^{n-j}(2j)!}{(n-j)!j!(2j+1-n)!} & \text{for } \dfrac{n-1}{2} \leqslant j \leqslant n \\ 0 & \text{for } 0 \leqslant j < \dfrac{n-1}{2}, \end{cases} \tag{8}$$

it is possible to write

$$\prod_{j=1}^{N} q_{n_j}(a_j u) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_N=0}^{n_N} \left(\prod_{j=1}^{N} \alpha_{k_j}^{(n_j)} a_j^{k_j} \right) \sum_{i=0}^{k_1+\cdots+k_N} \delta_i^{(k_1+\cdots+k_N)} q_i(u),$$

which gives an expression for β_j in (6), but because of the varying sign of $\delta_j^{(n)}$, it is not possible to see directly that $\beta_j \geqslant 0$.

If $a_j = 1/N$ and $n_1 = \cdots = n_N = 1$, i.e. the case of $f = 3$ degrees of freedom where $q_1(u) = 1 + u$, this formula simplifies to the expression given in Cufaro Petroni (2007, proposition 5.2). It is claimed that the expression is positive, but no convincing argument is given.

4. Second conjecture: the asymptotic behavior of the Student process

A second property studied by Cufaro Petroni is the asymptotic behavior of the distribution of the random walk Z_N; in the case of $f = 3$ degrees of freedom $\left(\nu = \frac{3}{2}\right)$, he obtains the following result (Cufaro Petroni 2007, proposition 5.1): for all $c > 0$,

$$p_{\frac{3}{2}}^{*c}(x) \sim \frac{2c}{\pi x^4}, \qquad x \to +\infty.$$

We now provide an extension of this result to any value $f = 2\nu$ ($\nu > 0$) of degrees of freedom. Cufaro Petroni's argument is via Fourier analysis. This argument becomes very technical if one tries to generalize it to arbitrary degrees of freedom. Our proof is based on results about subexponential distributions.

Theorem 2. *For any $c > 0$ and $\nu > 0$, the density of the c-fold convolution of the d-variate Student t-distribution behaves asymptotically as*

$$p_\nu^{*c}(\mathbf{x}) \sim \frac{cA_{d,\nu}}{|\mathbf{x}|^{2\nu+d}}, \qquad |\mathbf{x}| \to +\infty.$$

Proof. The proof is based on a series of lemmas given in the last section. The d-variate Student t-distribution is subordinated to the d-variate Gaussian semigroup

$$g_t(\mathbf{x}) = (4\pi t)^{-d/2} \exp\left(-\frac{|\mathbf{x}|^2}{4t}\right), \qquad t > 0, \qquad \mathbf{x} \in \mathbb{R}^d$$

by the inverse Gamma density, i.e.

$$p_\nu(\mathbf{x}) = \int_0^{+\infty} g_t(\mathbf{x})\, \mathrm{d}H_\nu(t),$$

where $H_\nu(t)$ is the inverse Gamma distribution with density

$$h_\nu(t) = C_\nu \exp\left(-\frac{1}{4t}\right) t^{-\nu-1}, \qquad t > 0, \qquad C_\nu = \frac{1}{2^{2\nu}\Gamma(\nu)}.$$

From this representation we deduce in lemma 1 the same representation for the c-fold convolution power of the Student t-density, namely

$$p_\nu^{*c}(\mathbf{x}) = \int_0^{+\infty} g_t(\mathbf{x})\, \mathrm{d}H_\nu^{*c}(t).$$

We note that this property is very general in the sense that it holds for any infinitely divisible probability distribution $\mathrm{d}H(t)$ on $[0, \infty[$.

The next step of the proof is the derivation of the asymptotic behavior of the c-fold convolution power of the inverse Gamma density $h_\nu(t)$: by lemma 2, this reads

$$h_\nu^{*c}(t) \sim cC_\nu t^{-\nu-1}, \qquad t \to +\infty.$$

Finally, we show in lemma 4 that this asymptotic behavior implies, by subordination to the Gaussian semigroup, the desired asymptotic behavior of the c-fold Student t-convolution. □

As a consequence of this theorem, we deduce the following.

Corollary 1. *In the case where the number of degrees of freedom $2\nu = 2n + 1$ is an odd integer and with integer N, the coefficient $\beta_n^{(n,N)}$ in (5) reads*

$$\beta_n^{(n,N)} = \frac{1}{N^{2n}}.$$

Proof. Since the coefficients $\beta_k^{(n,N)}$ do not depend on the dimension d, we consider the case $d = 1$. The function v in (5) is the density of the normalized random walk $\frac{1}{N}\sum_{i=1}^{N} X_i$ and thus is written as

$$v(x) = Np_{n+\frac{1}{2}}^{*N}(Nx).$$

By theorem 2,

$$v(x) \sim N^2 \frac{A_{n+\frac{1}{2}}}{(Nx)^{2n+2}} = \frac{A_{n+\frac{1}{2}}}{N^{2n}} x^{-2n-2}, \qquad x \to +\infty.$$

Since each Student t-distribution $p_{k+\frac{1}{2}}$ in (5) has asymptotic behavior

$$p_{k+\frac{1}{2}}(x) \sim A_{k+\frac{1}{2}} x^{-2k-2},$$

we deduce that

$$v(x) \sim \beta_n^{(n,N)} A_{n+\frac{1}{2}} x^{-2n-2}.$$

Identification of the two equivalents yields the result. □

5. Third conjecture: non-integer sampling time and odd degrees of freedom

In this section, we prove by contradiction the following result.

Theorem 3. *For all $c > 0$, $c \notin \mathbb{N}$ and $\nu = n + \frac{1}{2}$, $n \in \mathbb{N}$, the univariate density p_ν^{*c} cannot be expanded as*

$$p_{n+\frac{1}{2}}^{*c}(x) = \sum_{j=0}^{+\infty} \beta_j \frac{1}{c} p_{j+\frac{1}{2}}\left(\frac{x}{c}\right), \tag{9}$$

with parameters $\beta_j \geqslant 0$.

Proof. We remark that integrating equality (9) over $\mathbb{R}$ yields $\sum_{j=0}^{+\infty} \beta_j = 1$ so that the sequence (β_k) is summable. The Fourier transform of (9) reads

$$k_{n+\frac{1}{2}}^{c}(u) = \sum_{j=0}^{+\infty} \beta_j \exp(-cu) q_j(cu), \qquad u > 0$$

where q_j is the Bessel polynomial of degree j. Thus, by lemma 6, the sum $\sum_{j=0}^{+\infty} \beta_j q_j(cu)$ is an entire function, so that the function $k_{n+\frac{1}{2}}^{c}(u)$ extends to an entire function. But

$$k_{n+\frac{1}{2}}^{c}(u) = \exp(-cu)[q_n(u)]^c,$$

and since c is not an integer, the function q_n^c is not holomorphic at any of the complex roots of q_n, which concludes the proof. □

6. Lemmas for the proof of theorems 2 and 3

Lemma 1. *The c-fold convolution of the density p_ν reads, for all $c > 0$ and $\nu > 0$,*

$$p_\nu^{*c}(\mathbf{x}) = \int_0^{+\infty} g_t(\mathbf{x})\, \mathrm{d}H_\nu^{*c}(t).$$

Proof. Let us consider the d-dimensional Fourier transform $\mathcal{F}$ of the right-hand side

$$\mathcal{F}\left[\int_0^{+\infty} g_t(\mathbf{x})\,\mathrm{d}H_\nu^{*c}(t)\right](\mathbf{y}) = \left[\int_0^{+\infty} \mathcal{F}[g_t(\mathbf{x})]\,\mathrm{d}H_\nu^{*c}(t)\right](\mathbf{y})$$
$$= \int_0^{+\infty} \exp(-t|\mathbf{y}|^2)\,\mathrm{d}H_\nu^{*c}(t).$$

This last integral is nothing but the Laplace transform $\mathcal{L}$ of H_ν^{*c} evaluated at $|\mathbf{y}|^2$, and thus coincides with

$$\mathcal{L}(H_\nu)^c(|\mathbf{y}|^2) = \left(\int_0^{+\infty} \exp(-t|\mathbf{y}|^2)\,\mathrm{d}H_\nu(t)\right)^c = (\mathcal{F}(p_\nu(\mathbf{x})))^c(\mathbf{y}).$$

The result follows by considering the inverse Fourier transform. □

Lemma 2. *For all $c > 0$ and $\nu > 0$, the c-fold convolution of the inverse Gamma density has asymptotical behavior*

$$h_\nu^{*c}(t) \sim cC_\nu t^{-\nu-1}, \qquad t \to +\infty.$$

Proof. Since

$$h_\nu(t) = C_\nu \exp\left(-\frac{1}{4t}\right) t^{-\nu-1} \sim C_\nu t^{-\nu-1}, \qquad t \to +\infty,$$

the tail function $\bar{H}_\nu(t) = 1 - H_\nu(t)$ of the inverse Gamma distribution has the asymptotic behavior

$$\bar{H}_\nu(t) \sim \frac{C_\nu}{\nu} t^{-\nu}, \qquad t \to +\infty.$$

This tail function is thus regularly varying and, by Feller (1950, p 278),

$$\bar{H}_\nu^{*2}(t) \sim \frac{2C_\nu}{\nu} t^{-\nu}$$

where $\bar{H}_\nu^{*2}$ is the tail of H_ν^{*2}, so that the inverse Gamma distribution is subexponential in the sense of Chistyakov, cf Chistyakov (1964). Since it is moreover infinitely divisible, we deduce by Embrechts *et al* (1979, corollary 1, p 340) that for all $c > 0$

$$\bar{H}_\nu^{*c}(t) \sim \frac{cC_\nu}{\nu} t^{-\nu}, \qquad t \to +\infty.$$

By lemma 3 the density $h_\nu^{*c}(t)$ is ultimately decreasing, so the result follows by the application of the monotone density theorem (Bingham *et al* 1987). □

Lemma 3. *The c-fold convolution of the inverse Gamma density is ultimately decreasing.*

Proof. The inverse Gamma density $h_\nu(t)$ is a generalized Gamma convolution, and so is the convolution power h_ν^{*c}, cf Steutel and van Harn (2004, p 350). Since its left extremity is 0, we deduce from Steutel and van Harn (2004, proposition 5.5) that it is unimodal, and thus ultimately decreasing. □

Lemma 4. *The asymptotic behavior of the c-fold Student t-convolution is*

$$p_\nu^{*c}(\mathbf{x}) \sim \frac{cA_{d,\nu}}{|\mathbf{x}|^{2\nu+d}}, \qquad |\mathbf{x}| \to \infty.$$

Proof. Since by lemma 2

$$h_\nu^{*c}(t) \sim cC_\nu t^{-\nu-1},$$

for any a and b such that $a < cC_\nu < b$, there exists $t_0 > 0$ such that for all $t > t_0$,

$$at^{-\nu-1} \leqslant h_\nu^{*c}(t) \leqslant bt^{-\nu-1}.$$

From

$$\int_0^{+\infty} g_t(\mathbf{x})h_\nu^{*c}(t)\,\mathrm{d}t = \int_0^{t_0} g_t(\mathbf{x})h_\nu^{*c}(t)\,\mathrm{d}t + \int_{t_0}^{+\infty} g_t(\mathbf{x})h_\nu^{*c}(t)\,\mathrm{d}t,$$

it follows that

$$\int_0^{t_0} g_t(\mathbf{x})h_\nu^{*c}(t)\,\mathrm{d}t + \int_{t_0}^{+\infty} g_t(\mathbf{x})\frac{a}{t^{\nu+1}}\,\mathrm{d}t \leqslant \int_0^{+\infty} g_t(\mathbf{x})h_\nu^{*c}(t)\,\mathrm{d}t$$
$$\leqslant \int_0^{t_0} g_t(\mathbf{x})h_\nu^{*c}(t)\,\mathrm{d}t + \int_{t_0}^{+\infty} g_t(\mathbf{x})\frac{b}{t^{\nu+1}}\,\mathrm{d}t.$$

But the integral $\int_0^{t_0} g_t(\mathbf{x})h_\nu^{*c}(t)\,\mathrm{d}t$ is $o(|\mathbf{x}|^{-2\nu-d})$ for $|\mathbf{x}| \to \infty$ because

$$\int_0^{t_0} g_t(\mathbf{x})h_\nu^{*c}(t)\,\mathrm{d}t = \int_0^{t_0} \frac{1}{(4\pi t)^{\mathrm{d}/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4t}\right) h_\nu^{*c}(t)\,\mathrm{d}t$$
$$\leqslant \exp\left(-\frac{|\mathbf{x}|^2}{4t_0}\right)\int_0^{t_0} \frac{1}{(4\pi t)^{\mathrm{d}/2}} h_\nu^{*c}(t)\,\mathrm{d}t.$$

A simple computation gives

$$\int_{t_0}^{+\infty} g_t(\mathbf{x})t^{-\nu-1}\,\mathrm{d}t = |\mathbf{x}|^{-2\nu-d}\frac{2^{2\nu}}{\pi^{\mathrm{d}/2}}\int_0^{\frac{|\mathbf{x}|^2}{4t_0}} \exp(-u)u^{\nu+\frac{\mathrm{d}}{2}-1}\,\mathrm{d}u \sim |\mathbf{x}|^{-2\nu-d}\frac{2^{2\nu}}{\pi^{d/2}}\Gamma\left(\nu+\frac{d}{2}\right)$$

hence

$$\limsup_{|\mathbf{x}|\to\infty} |\mathbf{x}|^{2\nu+d}\int_0^{+\infty} g_t(\mathbf{x})h_\nu^{*c}(t)\,\mathrm{d}t \leqslant \limsup_{|\mathbf{x}|\to\infty} |\mathbf{x}|^{2\nu+d}\int_{t_0}^{+\infty} g_t(\mathbf{x})\frac{b}{t^{\nu+1}}\,\mathrm{d}t$$
$$= \frac{b2^{2\nu}}{\pi^{d/2}}\Gamma\left(\nu+\frac{d}{2}\right)$$

and

$$\liminf_{|\mathbf{x}|\to\infty} |\mathbf{x}|^{2\nu+d}\int_0^{+\infty} g_t(\mathbf{x})h_\nu^{*c}(t)\,\mathrm{d}t \geqslant \liminf_{|\mathbf{x}|\to\infty} |\mathbf{x}|^{2\nu+d}\int_{t_0}^{+\infty} g_t(\mathbf{x})\frac{a}{t^{\nu+1}}\,\mathrm{d}t$$
$$= \frac{a2^{2\nu}}{\pi^{d/2}}\Gamma\left(\nu+\frac{d}{2}\right),$$

so that finally

$$\int_0^{+\infty} g_t(\mathbf{x})h_\nu^{*c}(t)\,\mathrm{d}t \sim c\frac{2^{2\nu}}{\pi^{d/2}}\Gamma\left(\nu+\frac{d}{2}\right)C_\nu|\mathbf{x}|^{-2\nu-d} = cA_{d,\nu}|\mathbf{x}|^{-2\nu-d}.$$

□

Lemma 5. *For fixed k, the coefficients $\alpha_k^{(n)}$ of the Bessel polynomial of degree $n \geqslant k$ are increasing in n and*

$$\lim_{n\to+\infty} \alpha_k^{(n)} = \frac{1}{k!}.$$

Proof. From (4) we get

$$\alpha_k^{(n)} = \frac{1}{k!}\prod_{j=1}^{k-1}\frac{n-j}{n-\frac{j}{2}} \leqslant \frac{1}{k!},$$

where each of the $(k-1)$ terms of the product

$$\frac{n-j}{n-\frac{j}{2}} = 1 - \frac{\frac{j}{2}}{n-\frac{j}{2}}$$

is increasing and converges to 1 with n. □

Lemma 6. *Consider the infinite series* (S) *equal to*

$$\sum_{k=0}^{+\infty} c_k q_k(z),$$

where $q_k(z)$ *is the kth Bessel polynomial and* $c_k \in \mathbb{C}$. *Then the three following assertions are equivalent:*

(i) (S) *is absolutely convergent for* $z = 0$,
(ii) the sequence (c_k) *is absolutely summable,*
(iii) the series (S) *converges absolutely and uniformly on any compact subset of* $\mathbb{C}$.

Proof. (i) ⇒ (ii) since $q_k(0) = 1$.
(ii) ⇒ (iii) since

$$|c_n q_n(z)| \leqslant |c_n| q_n(|z|) \leqslant |c_n| \sum_{k=0}^{n} \frac{|z|^k}{k!} \leqslant |c_n| \sum_{k=0}^{+\infty} \frac{|z|^k}{k!} \leqslant |c_n| \exp(K)$$

for some constant K. The first inequality holds since all the coefficients $\alpha_k^{(n)}$ of the Bessel polynomial q_n are positive; the second inequality is a consequence of the majorization $\alpha_k^{(n)} \leqslant \frac{1}{k!}$ proved in lemma 5; the third inequality is straightforward, and the last inequality ensues from the assumption that z belongs to a compact subset of $\mathbb{C}$. Since the sequence (c_n) is assumed to be absolutely summable, the absolute and uniform convergence of (S) is a direct consequence of the above majorization.
(iii) ⇒ (i) trivially. □

7. Conclusion

In this paper, we have proved two of the conjectures as expressed by Cufaro Petroni (2007), and disproved a simple version of the third conjecture. We note that these results extend naturally to d-dimensional Student t-vectors with correlated components, with density

$$p_\nu(\mathbf{x}) = \frac{A_{d,\nu}}{|K|}(1 + \mathbf{x}^t K^{-1}\mathbf{x})^{-(\nu+\frac{d}{2})}$$

and the characteristic function

$$\varphi(\mathbf{u}) = k_\nu(\sqrt{\mathbf{u}^t K \mathbf{u}})$$

where K is a symmetric and positive definite matrix. Theorem 1 holds unchanged and the asymptotic result of theorem 2 still holds by replacing constant $A_{d,\nu}$ by $\frac{A_{d,\nu}}{|K|^{\frac{1}{2}}}$.

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